EFFECT OF HEATING ON PARAMETRIC EXCITATION OF WAVES ON THE SURFACE OF A LIQUID

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The threshold of excitation and the wave numbers of parametric waves on the surface of a layer of viscous incompressible liquid undergoing harmonic vibrations along a vertical axis are determined for an equilibrium temperature gradient.

We consider a horizontal layer of an incompressible liquid with a free upper boundary, bounded below by a solid isothermal plane maintained at a constant temperature Θ_0. A Cartesian coordinate system is chosen so that its x,y plane coincides with the unperturbed surface of the liquid, and its z axis is vertical. The layer moves vertically with a frequency ω according to the law $a\cos\omega t$, where the amplitude of vibration $a < 1$.

In the reference system fixed in the layer the convective equations in the Boussinesq approximation are written in standard form [1-3] with the effective force of gravity

$$\rho\vec{g}(1-\eta\cos\omega t), \tag{1}$$

where $\eta = a\omega^2/g$ is the dimensionless amplitude of modulation, and $\vec{g} = \{0, 0, -g\}$.

The uniform effective force (1) does not disturb equilibrium, giving rise only to pressure pulsations. Then the equilibrium conditions are

$$\vec{v}_0 = 0;\quad \nabla T_0 = G\frac{\vec{g}}{g};\quad p_0 = -\rho g z\left(1-\frac{\beta G z}{2}\right)(1-\eta\cos\omega t). \tag{2}$$

Here the pressure p and the temperature T are measured from their values at the surface, and $G = \theta_0/h$, where h is the thickness of the layer.

To investigate the stability of this equilibrium we introduce perturbations of the velocity, temperature, and pressure:

$$\vec{v}\to\vec{v}_0+\vec{v}';\quad T\to\Theta_0+\vartheta';\quad p\to p_0+p'$$

and take as units of length, time, frequency, pressure, and temperature $(\alpha/\rho g)^{1/2}$, $(\alpha/\rho g^3)^{1/4}$, $(\rho g^3/\alpha)^{1/4}$, $(\alpha g/\rho)^{1/4}$, $(\alpha\rho g)^{1/2}$, and $G(\alpha/\rho g)^{1/2}$, respectively. Omitting primes, we obtain the following system of linearized equations for the perturbations:

$$\begin{cases}\dfrac{\partial\vec{v}}{\partial t} = -\nabla p + \dfrac{1}{A}\nabla^2\vec{v} + q\left(1-\eta\cos\omega t\right)\dfrac{\vec{g}}{g}\,\vartheta,\\[2mm] P\left(\dfrac{\partial\vartheta}{\partial t} - w\right) = \dfrac{1}{A}\nabla^2\vartheta,\ \nabla\vec{v}=0,\ \nabla=\left\{\dfrac{\partial}{\partial x},\ \dfrac{\partial}{\partial y},\ \dfrac{\partial}{\partial z}\right\},\end{cases} \tag{3}$$

where

$$A=\frac{1}{\nu}(\alpha^3/\rho^3 g)^{1/4},\quad q = G\beta(\alpha/\rho g)^{1/2},\quad P=\frac{\nu}{\varkappa},\quad \vec{v}=\{u,\ v,\ w\}.$$

Assuming that the displacement ζ of the surface from its equilibrium position is small, we obtain the following equations for z = 0 [3]:

$$\frac{\partial\zeta}{\partial t} = w,$$

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$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=0,\quad \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=0,$$
$$p=\left[1-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\eta\cos\omega t\right]\zeta+\frac{2}{A}\frac{\partial w}{\partial z}. \qquad (4)$$

Assuming that the temperature of the free surface is maintained constant, the temperature perturbations vanish at the surface, and to first-order terms in ζ we have [2]

$$\vartheta=\zeta. \qquad (5)$$

On the bottom, i.e., for $z = -H$, where H is the dimensionless thickness of the layer, we have

$$u=0,\quad v=0,\ w=0,\quad \vartheta=0. \qquad (6)$$

Taking Fourier transforms with respect to the variables x and y, and the Laplace transform with respect to the time t, eliminating the x and y components of the velocity and pressure, and using the fact that the perturbations of velocity and displacement are zero at zero time, we obtain instead of (3)-(6) the following expressions:

$$\left(\frac{d^2}{dz^2}-k^2\right)^2W(s)=As\left(\frac{d^2}{dz^2}-k^2\right)W(s)-Aqk^2\left\{\Theta(s)-\frac{\eta}{2}[\Theta(s+i\omega)+\Theta(s-i\omega)]\right\},$$
$$P[s\Theta(s)-W(s)]=\frac{1}{A}\left(\frac{d^2}{dz^2}-k^2\right)\Theta(s),\ k^2=k_x^2+k_y^2. \qquad (7)$$

At $z = 0$ we have

$$sZ(s)=W(s),\ \left(\frac{d^2}{dz^2}+k^2\right)W(s)=0,\ \Theta(s)=Z(s), \qquad (8)$$

$$\frac{1}{k}\left[s-\frac{1}{A}\left(\frac{d^2}{dz^2}-3k^2\right)\right]\frac{dW(s)}{dz}+(k^3+k)Z(s)=\frac{k\eta}{2}[Z(s+i\omega)+Z(s-i\omega)]. \qquad (9)$$

At $z = -H$ we have

$$W(s)=0,\ \frac{dW(s)}{dz}=0,\ \Theta(s)=0. \qquad (10)$$

We consider the limiting cases of small and large Prandtl numbers, for example, liquid metals and oils, respectively. For simplicity, we consider large depths, i.e., $H \gg (\alpha/\rho g)^{1/2}$.

1. For small Prandtl numbers the second of Eqs. (7) takes the form

$$\frac{d^2\Theta(s)}{dz^2}-k^2\Theta(s)=0. \qquad (1.1)$$

Then using (8) and (10) we obtain

$$\Theta(s)=Z(s)\exp(kz), \qquad (1.2)$$
$$W(s)=\frac{kqz}{2s}\left\{Z(s)-\frac{\eta}{2}[Z(s+i\omega)+Z(s-i\omega)]\right\}\exp(kz)+C_1\exp(kz)+C_2\exp(\sqrt{k^2+As}\,z),$$

where

$$C_1=sZ(s)+\frac{2k^2}{A}Z(s)+\frac{k^2q}{As^2}\left\{Z(s)-\frac{\eta}{2}[Z(s+i\omega)+Z(s-i\omega)]\right\},$$
$$C_2=-\frac{2k^2}{A}Z(s)-\frac{k^2q}{As^2}\left\{Z(s)-\frac{\eta}{2}[Z(s+i\omega)+Z(s-i\omega)]\right\}.$$

Substituting (1.2) into (9) and taking the inverse Laplace transform, we obtain the following expression for $\zeta(t)$ to an accuracy $1/A$ and q:

$$\frac{d^2\zeta}{dt^2}+2\delta\frac{d\zeta}{dt}+(\Omega_0^2-k\eta_1\cos\omega t)\zeta+\varepsilon_1\int_0^t\zeta(\tau)\,d\tau=0, \qquad (1.3)$$

where

$$\delta=\frac{2k^2}{A},\ \Omega_0^2=k^3+k+\frac{q}{2},\quad \varepsilon_1=\frac{\delta q}{2},\quad \eta_1=\eta\left(1+\frac{q}{2k}\right).$$

For $\delta = 0$ this equation goes over into Mathieu's equation [4]. Solving (1.3) by the method of averaging [5] in the neighborhood of the main resonance, we obtain the following equation for the bounds of the first region of stability:

$$\frac{\eta_1^2}{4\omega^2(\delta-2\varepsilon_1)^2}-\frac{(\Omega_0^2-\omega^2/4)^2}{\omega^2(\delta-2\varepsilon_1)^2}=1, \tag{1.4}$$

from which it follows that parametric waves with the frequency $\omega/2$ and wave number k arise on the surface of the liquid for

$$\eta\geqslant\frac{4k\omega}{A}\left(1-\frac{q}{2k}-\frac{q}{\omega^2}\right). \tag{1.5}$$

In this case waves with wave number

$$k=k_0-\frac{q}{2(1+3k_0^2)}+O(q^2), \tag{1.6}$$

are most easily excited, where

$$k_0=\sqrt[3]{\frac{\omega^2}{8}+\sqrt{\frac{1}{27}+\frac{\omega^4}{64}}}+\sqrt[3]{\frac{\omega^2}{8}-\sqrt{\frac{1}{27}+\frac{\omega^4}{64}}}.$$

2. For large Prandtl numbers the second of Eqs. (7) takes the form

$$s\Theta(s)=W(s). \tag{2.1}$$

Substituting (2.1) into the first of Eqs. (7), solving it with boundary conditions (8) and (10), taking the inverse Laplace transform, and neglecting terms of the order ηq, we obtain for the displacement of the surface from the equilibrium position an integrodifferential equation analogous to (1.3) with η and $2\varepsilon_1$ replacing η_1 and ε_1.

Then parametric waves with the frequency $\omega/2$ and wave number k appear on the surface for

$$\eta\geqslant\frac{4k\omega}{A}\left(1-\frac{2q}{\omega^2}\right). \tag{2.2}$$

Thus, the stability of the system is decreased ($q > 0$) or increased ($q < 0$) depending on the conditions of heating.

NOTATION

x, y, z, Cartesian coordinates; t, time; ω, frequency; a, amplitude of vibration; ρ, density of liquid; g, acceleration due to gravity; ν, coefficient of kinematic viscosity; β, volume coefficient of expansion; χ, thermal diffusivity; η, η_1, dimensionless amplitudes of modulation; $\vec{v} = \{u, v, w\}$, velocity vector; T, Θ_0, ϑ, temperatures; p, pressure; G, temperature gradient; h, H, depth of layer; ζ, displacement of surface from equilibrium position; P, Prandtl number; A, q, small parameters; δ, ε, dissipative parameters; k_j, wave number along j-th axis; Z(s), Θ(s), W(s), Fourier–Laplace transforms for the displacement of the surface from its equilibrium position, perturbations of the temperature, and of the z component of velocity; α, surface tension.

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